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Potential symmetries and direct reduction methods of order two

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Abstract. For partial differential equations written in conservative form a remarkable link between potential symmetries and direct reduction methods of order two is enlightened.

1. Introduction

Let us consider a partial differential equation of the form

$$\Delta(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0 \tag{1}$$

where x and t are independent variables and $u = f(x, t)$ the dependent variable. We say that (1) admits a *direct reduction of order* n if there exists functions $z = \zeta(x, t)$ and $u = (U(x, t, w_1(z), \ldots, w_n(z))$ such that

$$u = U(x, t, w_1(\zeta(x, t)), \ldots, w_n(\zeta(x, t))) \tag{2}$$

reduces (1) to a coupled system of n distinct differential equations for $w_1(z), \ldots, w_2(z)$.

If $n = 1$ then from (2) we recover the Clarkson–Kruskal(–Lou) ansatz [1, 2], which has been shown by Pucci [3] to be a particular case of the non-classical method of Bluman–Cole [4]. For nice overviews on the subject of non-classical weak (or conditional) symmetries and direct methods we refer to [5] and [6].

Recently, Olver [7] has shown that there is a one-to-one correspondence between direct reduction ansatz (2) and nth-order differential constraints of the form

$$\boldsymbol{v}^n(u) = \Phi(x, t, u, \boldsymbol{v}(u), \ldots, \boldsymbol{v}^{n-1}(u)) \tag{3}$$

where

$$\boldsymbol{v} = \tau(x, t)\partial_t + \xi(x, t)\partial_x \tag{4}$$

is a vector field, $\boldsymbol{v}^2(u) = \boldsymbol{v}(\boldsymbol{v}(u))$ and so on.

In [7] it is shown that the ansatz (2) always reduces (1) to a coupled system on n distinct ordinary differential equations for $w_1(z), \ldots, w_n(z)$ *if and only if* the overdetermined system of partial differential equations defined by (1) and (3) is compatible.

Since in the case $n = 2$ we have

$$\begin{aligned}\boldsymbol{v}^2(u) &= \tau^2 u_{tt} + 2\tau\xi u_{xt} + \xi^2 u_{xx} + (\tau\tau_t + \xi\tau_x)u_t + (\tau\xi_t + \xi\xi_x)u_x \\ &= \Phi(x, t, u, \tau u_t + \xi u_x)\end{aligned} \tag{5}$$

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then we can always consider (2) as an invariant solution under the action of a non-classical generalized symmetry [8] for which (5) is the infinitesimal generator in evolutionary form [9].

The aim of this paper is to give another (and, we think, more useful) group theoretical interpretation of direct reductions of order two for a particular class of partial differential equations (1).

Bluman *et al* [10] have introduced the concept of potential symmetry for any differential equation which can be written as a conservation law. In the case considered here, this means that (1) can be written as

$$\mathrm{D}_x F(x,t,u,u_x,u_t) - \mathrm{D}_t G(x,t,u,u_x,u_t) = 0 \tag{6}$$

where D_x and D_t are the total derivative operators. Introducing an auxiliary potential variable $v = v(x,t)$ it is possible to form the potential system, $S = 0$,

$$\begin{aligned} & v_t - F(x,t,u,u_x,u_t) = 0 \\ & v_x - G(x,t,u,u_x,u_t) = 0 \end{aligned} \tag{7}$$

which is obviously equivalent to (6).

To compute the *classical* point symmetries of (7), [9, 11], we introduce the infinitesimal generator

$$\chi = \xi(x,t,u,v)\partial_x + \tau(x,t,u,v)\partial_t + \eta(x,t,u,v)\partial_u + \phi(x,t,u,v)\partial_v$$

and its first-order prolongation

$$\chi^1 = \chi + \eta^x\partial_{u_x} + \eta^t\partial_{u_t} + \phi^x\partial_{v_x} + \phi^t\partial_{v_t} \tag{8}$$

where

$$\begin{aligned} \eta^x &= \mathrm{D}_x\eta - u_x\mathrm{D}_x\xi - u_t\mathrm{D}_x\tau \qquad & \eta^t &= \mathrm{D}_t\eta - u_x\mathrm{D}_t\xi - u_t\mathrm{D}_t\tau \\ \phi^x &= \mathrm{D}_x\phi - v_x\mathrm{D}_x\xi - v_t D_x\tau \qquad & \phi^t &= \mathrm{D}_t\phi - v_x\mathrm{D}_t\xi - v_t\mathrm{D}_t\tau. \end{aligned} \tag{9}$$

Considering the relation

$$\chi^1 S|_{S=0} = 0$$

we obtain the defining equations of the classical point symmetries admitted by (7). Any admitted symmetry with infinitesimal generator χ where ξ, τ or η depend on v is called potential symmetry of (6); potential symmetries are non-local symmetries.

Non-classical potential symmetries, [12], are simply obtained requiring that

$$\chi^1 S|_{S'=0} = 0$$

where $S' = 0$ is the overdetermined system obtained appending to $S = 0$ the invariant surface conditions

$$\eta - \xi u_x - \tau u_t = 0 \tag{10a}$$

$$\phi - \xi v_x - \tau v_t = 0. \tag{10b}$$

Since, in some cases, there are many ways to rewrite a given differential equation as a conservation law, potential symmetries depend on the chosen conservative form [13, 14]. For this reason, in general, it is very hard to characterize all the potential symmetry of a given equation.

In any case, [13], the similarity solutions associated with potential symmetries (classical or non-classical) are of the kind

$$u = U(x, t, z, w_1(z), w_2(z)) \tag{11a}$$

$$v = V(x, t, z, w_1(z), w_2(z)) \tag{11b}$$

$$G(x, t, z, w_1(z), w_2(z)) = 0 \tag{11c}$$

where the last equation defines implicitly the similarity variable z as a function of (x, t). It is possible to show [13], that (11a) introduced directly in equation (6) and not in system (7), as is usually done to find the similarity solutions, allows one to obtain a wider class of solutions of the original equation than the invariant one. Moreover, this wider class of solutions also satisfies a second-order equation

$$\eta^*(x, t, u, u_x, u_t, u_{xx}, u_{tx}, u_{tt}) = 0 \tag{12}$$

obtained by eliminating v from the corresponding invariant surface conditions (10). Indeed if we take the total derivatives of (10a) with respect to x and t we obtain

$$v_x = \frac{(\eta_x + \eta_u u_x - (\xi_x + \xi_u u_x)u_x - (\tau_x + \tau_u u_x)u_t - \xi u_{xx} - \tau u_{tx})}{\eta_v - \xi_v u_x - \tau_v u_t}$$

and

$$v_t = \frac{(\eta_t + \eta_u u_t - (\xi_t + \xi_u u_t)u_x - (\tau_t + \tau_u u_t)u_t - \xi u_{xt} - \tau u_{tt})}{\eta_v - \xi_v u_x - \tau_v u_t}$$

(we remember that $\xi_v^2 + \tau_v^2 + \eta_v^2 \neq 0$). Introducing these expressions into (10b) it is then possible to obtain a relation $H(x, t, v, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0$ which coupled with (10a) gives (12). This means, as already remarked in [13], that invariant solutions under the action of potential symmetries are also invariant solutions under non-classical generalized symmetries.

These facts seem to stress a deep link between potential symmetries and second-order differential constraints. Indeed, using the methods presented in [15] in this paper we show that for a partial differential equation (6) any compatible differential constraint of second order corresponds to a non-classical potential symmetry.

This result implies that the non-classical potential symmetries corresponding to a conservative form are all the possible potential symmetries of the given equation, so that all the problems relating to the choice of the conservative form can be bypassed. Moreover, also if at first sight potential symmetries can seem very special this is not the case since a large class of solutions can be obtained as invariant solutions under the action of these symmetries.

2. Potential symmetries and direct methods

Let us consider a second-order differential constraint

$$\boldsymbol{v}^2(u) = \Phi(x, t, u, \boldsymbol{v}(u)) \tag{13}$$

admitted by equation (6) and in correspondence with

$$\bar{\boldsymbol{v}} = \bar{\xi}(x, t)\partial_x + \bar{\tau}(x, t)\partial_t. \tag{14}$$

The fact that (13) is admitted by (6) means that the overdetermined system, Σ, composed by (6) and (13) is compatible, so therefore complete, and then any differential consequence of Σ is also algebraic consequence of this system.

Now let us consider the system $\bar{\Sigma}$ equivalent to Σ composed by the potential system (7) and the invariant surface conditions

$$\begin{aligned} \bar{\xi}u_x + \bar{\tau}u_t &= v \\ \bar{\xi}v_x + \bar{\tau}v_t &= \bar{\phi} \end{aligned} \tag{15}$$

where $\bar{\xi}$ and $\bar{\tau}$ are the same functions entering in the vector field (14) and $\bar{\phi} = \Phi(x, t, u, v)$ (see equation (5)).

Our claim is that

$$\bar{\chi} = \bar{\xi}(x, t)\partial_x + \bar{\tau}(x, t)\partial_t + v\partial_u + \Phi\partial_v$$

is a non-classical potential symmetry of (6). To this end, having $\eta = v$, it is sufficient to show that $\bar{\chi}$ is a non-classical point symmetry of the potential system, i.e. that

$$\bar{\chi}^1 S|_{\bar{\Sigma}} = 0 \tag{16}$$

but since $\bar{\Sigma}$ is complete (because so is Σ) this last condition is true if and only if $\bar{\chi}^1 S$ is a differential consequence of $\bar{\Sigma}$.

All the differential consequences of the first order of $\bar{\Sigma}$ must be combinations of

$$v_{tt} = \mathrm{D}_t F \qquad v_{tx} = \mathrm{D}_x F \tag{17}$$

$$v_{xt} = \mathrm{D}_t G \qquad v_{xx} = \mathrm{D}_x G \tag{18}$$

$$\eta^x = \xi u_{xx} + \tau u_{tx} \qquad \eta^t = \xi u_{xt} + \tau u_{tt} \tag{19}$$

$$\phi^x = \xi v_{xx} + \tau v_{tx} \qquad \phi^t = \xi v_{xt} + \tau v_{tt}. \tag{20a, b}$$

If we solve (19) for u_{xx} and u_{tt}, we obtain

$$u_{xx} = \frac{\eta^x - \tau u_{tx}}{\xi} \qquad u_{tt} = \frac{\eta^t - \xi u_{xt}}{\tau} \tag{21a, b}$$

so then substituting (21) and (18) into (20*a*), we have

$$\begin{aligned} \phi^x = {} & \xi(G_x + G_u u_x + G_{u_t} u_{tx}) + G_{u_x}(\eta^x - \tau u_{tx}) \\ & + \tau(G_t + G_u u_t + G_{u_x} u_{tx}) + G_{u_t}(\eta^t - \xi u_{tx}) \end{aligned} \tag{22}$$

and substituting (21) and (17) in (20*b*) we have

$$\begin{aligned} \phi^t = {} & \xi(F_x + F_u u_x + F_{u_t} u_{tx}) + F_{u_x}(\eta^x - \tau u_{tx}) \\ & + \tau(F_t + F_u u_t + F_{u_x} u_{tx}) + F_{u_t}(\eta^t - \xi u_{tx}). \end{aligned} \tag{23}$$

The relations (22) and (23) are exactly $\overline{\chi^1} S$ when we consider the system $\bar{\Sigma}$, and since these relations are obtained as linear combinations of the differential consequences of $\bar{\Sigma}$ we have shown that $\bar{\chi}$ is a potential symmetry of (6).

Now what about the *vice versa*? When given a potential symmetry is it possible to find a corresponding differential constraint of second order?

From the special form of $\bar{\chi}$ it seems that only very particular potential symmetries and differential constraints of second order are in correspondence. Indeed not only, as for the CK-direct method, do we have that our potential symmetries must have $\xi = \xi(x, t)$, $\tau = \tau(x, t)$, but the generator corresponding to u must be very special since from the previous result it seems that this infinitesimal generator must be exactly $\eta = v$. Given a potential symmetry where $\xi = \xi(x, t)$, $\tau = \tau(x, t)$, but $\eta = \eta(x, t, u, v)$ since the corresponding similarity solution is always of the kind (11*a*), and z is given explicitly, using Olver's results it is possible to find a differential second-order constraint generated by a vector field (4) in correspondence with this solution. Then, if we apply our result, it is always possible to find an equivalent (in the sense with the same invariant solutions) non-classical potential symmetry with $\eta = v$, the same ξ and τ, but different ϕ. In practice this can be done by using the method of characteristics backwards or as we illustrate in the following example.

Let us consider the Fokker–Planck equation

$$u_t = (xu_x)_x + u_{xx} \tag{24}$$

and the corresponding potential system

$$v_x = u \qquad v_t = u_x + xu. \tag{25}$$

Potential symmetries for these kind of equations have been investigated in [16] and [13] where it has been shown that

$$\xi = -x \qquad \tau = -1 \qquad \eta = u(x^2+2) + 2xv \qquad \phi = v(x^2+1) \tag{26}$$

is a (classical) potential symmetry for (24). The characteristic system related to the invariant surface conditions is

$$\begin{aligned} &xu_x + u_t + ux^2 + 2vx + 2u = 0 \\ &xv_x + v_t + v(x^2+1) = 0. \end{aligned} \tag{27}$$

The corresponding similarity reduction, for $u = u(x,t)$, is obtained as usual as

$$u = (h_2(z)x^{-2} - h_1(z)) \exp(-x^2/2) \tag{28}$$

where the similarity variable is $z(x,t) = x^{-1}\exp(t)$. The vector field in (26) is not as we require here since $\eta \neq v$. Using the method described in the introduction it is easy to show that the corresponding equation (12) for this potential symmetry is

$$\eta^* = u_{tt} + 2xu_{xt} + x^2u_{xx} + (2x^2+2)u_t + (2x^3+3x)u_x + (x^4+4x^2)u = 0.$$

Now by setting $\xi = -x$, $\tau = -1$ and $\eta = v$ the corresponding invariant surface condition is

$$-xu_x - u_t = v \tag{29}$$

and by total derivation of this relation it is possible to obtain

$$u_{tt} + 2xu_{xt} + x^2u_{xx} + xu_x = -xv_x - v_t.$$

By introducing the above expression and (29) in η^*, with some simple algebra we recognize that

$$\phi = (2x^2+2)v - (x^4+4x^2)u.$$

In such a way we have found the non-classical symmetry we were searching for.

In summary, direct reduction methods of order two are a particular case of non-classical potential symmetries and there is an exact equivalence between this direct reduction method and potential symmetries with infinitesimal generators which does not depend on the unknown functions.

Remark 1. The same arguments of the last section of [17] allows one here to enlarge the point of view considering the case where the similarity variable z is given implicitly by a relation of the kind (11*c*), and to show a perfect equivalence between non-classical potential symmetries and direct reduction methods of order two for partial differential equations written in conservative form.

Remark 2. From our results we deduce also that to compute non-classical potential symmetries of a given differential equation, not only is it sufficient to take only a conservative form, but it is also sufficient to search for generators in the form $\tau = 1$, $\xi = \xi(x, t, u, v)$, $\eta = v$, $\phi = \phi(x, t, u, v)$ or $\tau = 0$, $\xi = 1$, $\eta = v$, $\phi = \phi(x, t, u, v)$. The normalization $\tau = 1$ is the usual one suggested by the presence of the invariant surface conditions. The generality of the second normalization $\eta = v$ we have already discussed, but we remark again that it arises from the arbitrariness we have in the choice of the auxiliary potential variable [13, 16].

3. Concluding remarks

The main goal of this short paper was to stress the link between potential symmetries and reduction of order two for partial differential equations in two variables of second order. We have chosen this framework to present our results for the sake of simplicity and to allow an immediate comparison with the results previously presented in [7], but using the methods of [15] it is simple to generalize everything to partial differential equations in more variables and of any order. The ideas contained in [15] (we note that this paper has been previously a Mathematical Science Institute of the Cornell University Preprint in 1988) has been successfully used in several papers to include direct methods into a group theoretical framework [3, 18, 19].

We think the results here given are interesting for different reasons. Indeed, as we now also have a group theoretical framework for direct reduction methods of order two. The link between this direct method and potential symmetries allows one to show that the definition of non-classical potential symmetry is independent of the conservative form chosen to write the potential system equivalent to the given equation, and for their computation we can introduce a new normalization. As a last point we also have to note that despite their simplicity potential symmetries are related to a wide class of similarity reductions and, since equations of physics are often written in a conservative form (for example the balance laws of continuum mechanics), they are the natural way to study invariance.

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